Filter Banks – III

The Polyphase Representation

Dr. Yogananda Isukapalli

The Polyphase Representation

1. Basic idea
2. Types of polyphase representation
3. Polyphase implementations
   a) Decimation filters
   b) Interpolation filters
   c) Uniform DFT filter banks
4. Commutator models
5. Polyphase matrix
   a) Analysis bank
   b) Synthesis bank
6. Polyphase representation of M-channel QMF banks
7. Relation between polyphase matrix and AC matrix

1. Basic Idea

Polyphase representation leads to computationally efficient implementations of decimation/interpolation filters, as well as single and multirate filter banks

Consider a filter

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \qquad (1)$$

Separating odd and even numbered coefficients of h(n)

$$H(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h(2n+1) z^{-2n}$$

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2) \qquad (2)$$

where $E_0(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-n}$ and $E_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1) z^{-n}$

These representations hold whether H(z) is FIR or IIR; causal or noncausal

Example 1 - *Consider a FIR filter whose transfer function is*

$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$. It's polyphase representation is given as

$$H(z) = 1 + 3z^{-2} + 2z^{-1} + 4z^{-3}$$

i.e

$$H(z) = 1 + 3z^{-2} + z^{-1}(2 + 4z^{-2})$$

Therefore,

$$E_0(z^2) = 1 + 3z^{-2}, \qquad \mathrm{E}_1(z^2) = 2 + 4z^{-2} \qquad \textit{or}$$

$$E_0(z) = 1 + 3z^{-1}, \qquad \mathrm{E}_1(z) = 2 + 4z^{-1}$$

Example 2 - *Consider a IIR filter whose transfer function is* $H(z) = 1/(1 - \alpha z^{-1})$
This can be written as

$$H(z) = \frac{1 + \alpha z^{-1}}{(1 - \alpha z^{-1})(1 + \alpha z^{-1})} = \frac{1}{1 - \alpha^2 z^{-2}} + \frac{\alpha z^{-1}}{1 - \alpha^2 z^{-2}} \quad \textit{which implies}$$

$$E_0(z^2) = 1/(1 - \alpha^2 z^{-2}) \quad \text{and} \quad E_1(z^2) = \alpha/(1 - \alpha^2 z^{-2}$$

$$\textit{so } E_0(z) = 1/(1 - \alpha^2 z^{-1}) \quad \text{and} \quad E_1(z) = \alpha/(1 - \alpha^2 z^{-1})$$

2. Types of polyphase representation

Generalizing the above idea, for any integer M, H(z) can be decomposed as

$$H(z) = \sum_{n=-\infty}^{\infty} h(nM)z^{-nM} + z^{-1}\sum_{n=-\infty}^{\infty} h(nM+1)z^{-nM} + \ \ldots$$
$$\ldots + z^{-(M-1)}\sum_{n=-\infty}^{\infty} h(nM+M-1)z^{-nM} \qquad (3)$$

This can be compactly represented as

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M) \quad \text{(type 1 polyphase representation)} \qquad (4)$$

$$\text{where } E_l(z) = \sum_{n=-\infty}^{\infty} e_l(n) z^{-n}, \qquad (5)$$

$$\text{with } e_l(n) = h(Mn+l), \qquad 0 \le l \le M-1 \qquad (6)$$

Equation (4) is called the Type 1 polyphase representation and $E_l(z)$ the polyphase components of H(z). $E_l(z)$ depends on the choise of M.

A variation of (3) is given as

$$H(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_l(z^M) \quad \text{(type 2 polyphase representation) (7)}$$

The Type 2 polyphase components $R_l(z)$ are permutations of $E_l(z)$ and is given as

$$R_l(z) = E_{M-1-l}(z) \qquad (8)$$

3. Efficient structures for Decimation and Interpolation filters

Consider decimation of a signal by a factor 2.

$$y(n) = x(2n) \qquad (9)$$

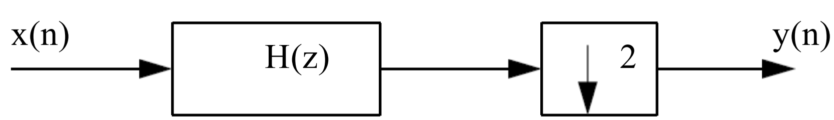


fig1. Decimation by 2

By representing H(Z) as in (2), the system can be redrawn as in *fig 2.a.*

From noble's identity 1, this can be again redrawn as in *fig 2.b*

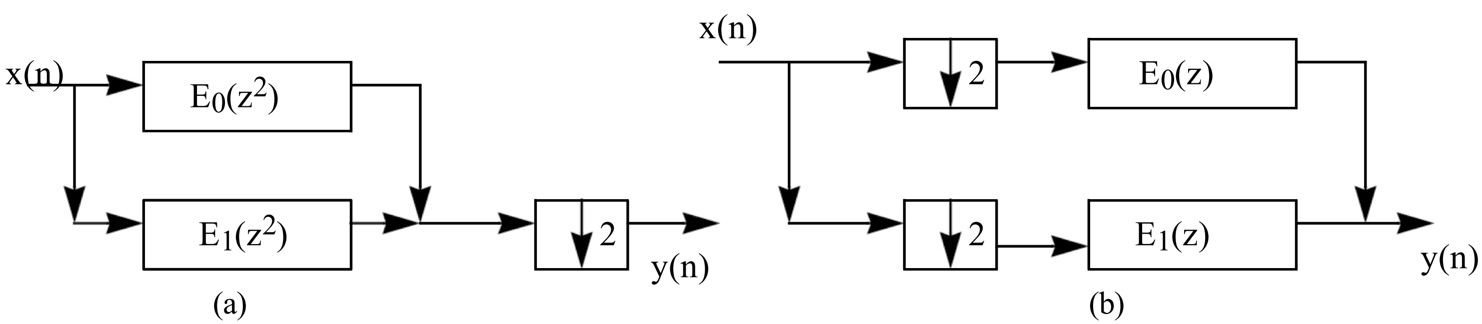
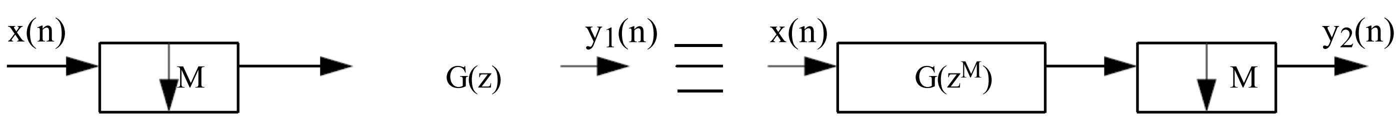


fig2. The decimation filter. (a) The polyphase implementation (b) moving the polyphase components

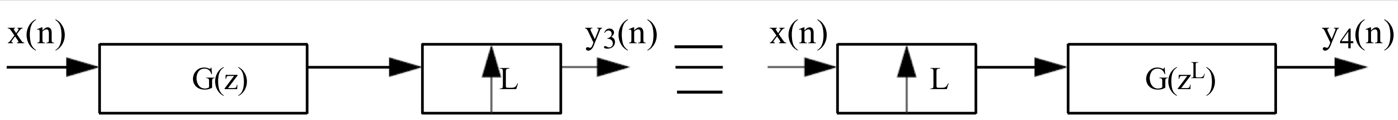
Note - The Noble's Identities

If a function G(z) is rational then

Identity 1



Identity 2



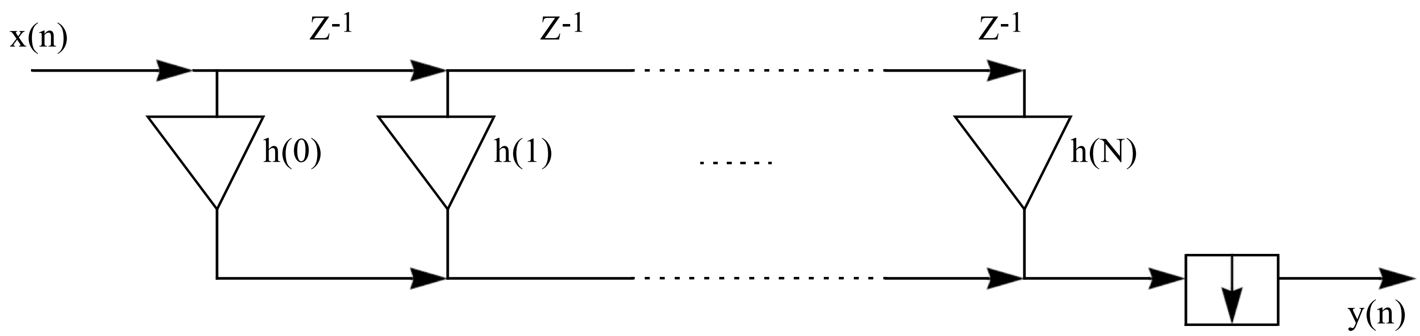


fig3 - Direct implementation of decimation

- For the direct form implementation only the even numbered output samples are computed in each time unit. This computation requires N+1 multiplications and N additions, where N is the order of the FIR filter.
- For the polyphase implementation, let n_o and n_1 be the orders of $E_0(z)$ and $E_1(z)$ (so that $N+1 = n_0+n_1+2$). So $E_l(z)$ requires n_l+1 multipliers and n_l additions. Therefore the total cost is again N+1 multipliers and N adders. But since $E_l(z)$ operates at half the rate of H(z), only a total of (N+1)/2 multiplications per unit time (MPU's) and N/2 additions per unit time (APU's) are required. The multipliers and adders in each of the filters $E_0(z)$ and $E_1(z)$ now have two units of time available for doing their work, and they are continually operative unlike the direct form implementation which is idle during odd instants of time

- A M-fold decimation filter can be implemented with approximately M-fold reduction in the number of MPU's and APU's by using the polyphase structure. The Polyphase structure has complexity (N+1)/M MPU's and N/M APU's.

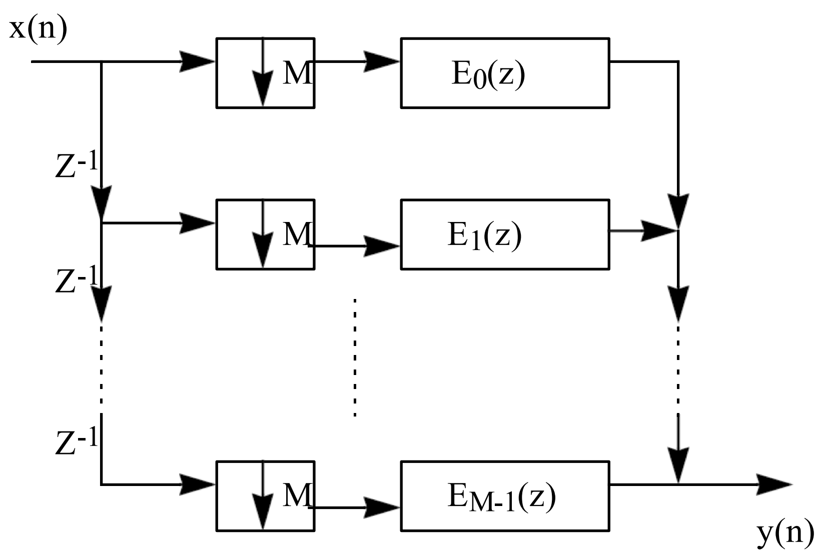


fig4. Polyphase implementation of M-fold decimation filter.

b) Interpolation filters

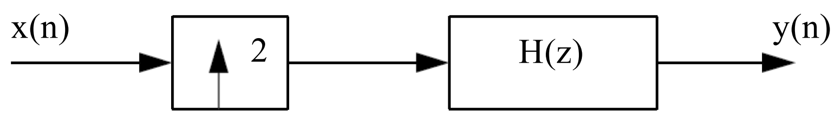


fig 5. Expanding by 2

Polyphase implementation of interpolation filter

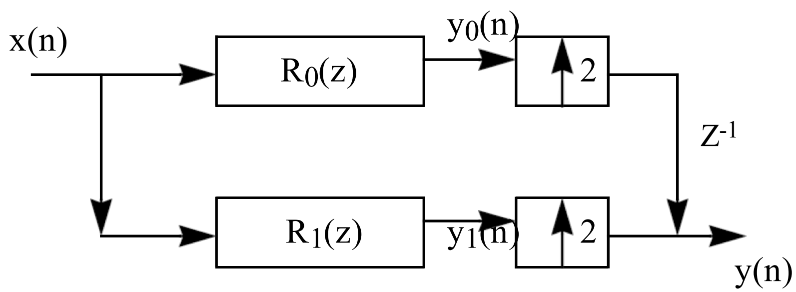


fig 6. Polyphase implementation of an interpolation filter

Polyphase representation (Type 2) of an interpolation filter

$$H(z) = R_1(z^2) + z^{-1} R_0(z^2) \qquad (10)$$

- A direct form implementation of H(z) is inefficient because, at most 50% of the input samples are nonzero and the remaining multipliers are resting. Those multipliers which are not resting are expected to complete their job in half unit of time because the outputs of the delay elements will change by that time. In a polyphase implementation Rl(z) are operating at the input rate, and none of the multipliers are resting. Each multiplier gets one unit of time to complete its task.The complexity of the system is (N+1) MPU's and N-1 APU's.
- Similarly for L-fold interpolation filters, the complexity is (N+1) MPU's and (N-L+1) APU's.

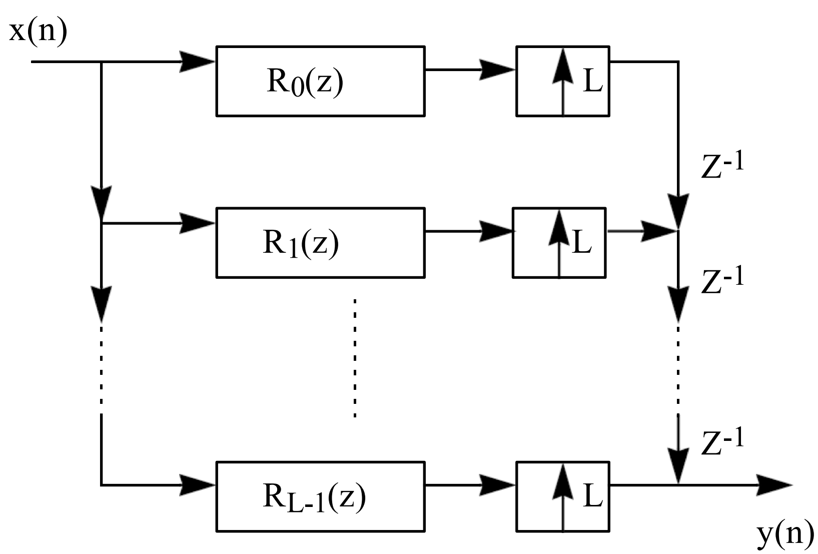


fig7. Polyphase implementations of L-fold interpolation filter

Linear phase FIR Decimation filters.

Let $H(z) = \sum_{n=0}^{N} h(n)z^{-n}$ where h(n) = h(N-n) (11)

Example 3 -

Let N = 4(even), $H(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$

Then it's polyphase representation is given as

$$H(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$$

i.e. $H(z) = 1 + 4z^{-2} + z^{-4} + 2z^{-1} + 2z^{-3}$

i.e. $H(z) = 1 + 4z^{-2} + z^{-4} + z^{-1}(2 + 2z^{-2})$

i.e $E_0(z^2) = 1 + 4z^{-2} + z^{-4}$ $E_1(z^2) = 2 + 2z^{-2}$ or

$E_0(z) = 1 + 4z^{-1} + z^{-2}$, $E_1(z) = 2 + 2z^{-1}$. So, each of the filters $E_0(z)$ and $E_1(z)$ has symmetric impulse response.

Consider N = 5(odd), $H(z) = 1 + 2z^{-1} + 4z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$, then it's polyphase representation is given as

$$H(z) = 1 + 4z^{-2} + 2z^{-4} + 2z^{-1} + 4z^{-3} + z^{-5}$$

i.e. $H(z) = 1 + 4z^{-2} + 2z^{-4} + z^{-1}(2 + 4z^{-2} + z^{-4})$

which implies $E_0(z^2) = 1 + 4z^{-2} + 2z^{-4}$ $E_1(z^2) = 2 + 4z^{-2} + z^{-4}$

Therefore,

$E_0(z) = 1 + 4z^{-1} + 2z^{-2}$, $E_1(z) = 2 + 4z^{-1} + z^{-2}$.

Here the filters are not symmetric but they are mirror images of one another.

- From this example we can generalize that for a linear FIR decimated filter, if $E_0(z)$ and $E_1(z)$ are the Type 1 polyphase components then
  - (a) if N is even, then $e_0(n)$ and $e_1(n)$ are symmetric sequences and
  - (b) if N is odd, then $e_0(n)$ is the mirror image of $e_1(n)$.

- For a linear FIR decimated filter, we obtain a factor of two saving in multiplication rate when compared with a non-linear FIR filter.

c) Polyphase implementation of Uniform DFT filter banks.

we have $X_k(z) = H_k(z)X(z)$ where (12)

$$H_k(z) = H_0(zW^k) \quad \text{with} \tag{13}$$

$$H_0(z) = 1 + z^{-1} + \ldots + z^{-(M-1)} \tag{14}$$

so, we have a bank of M filters, each of which is a uniformly shifted version of $H_0(z)$.

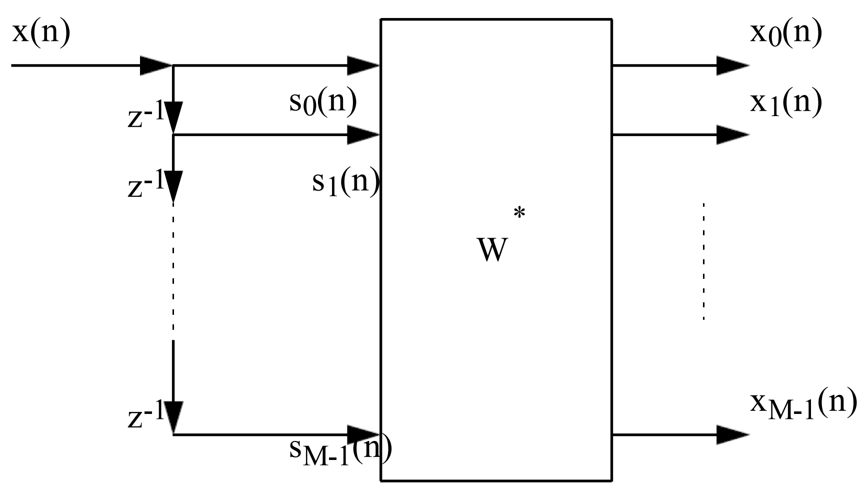


fig 8. Uniform DFT filter bank

Polyphase representation

Assume that the prototype $H_0(z)$ has been expressed as in 14. The k^{th} filter can then be expressed as $H_k(z) = H_0(zW^k) = \sum_{l=0}^{M-1} (z^{-1}W^{-k})^l E_l(z^M)$ and the output $X_k(z)$ can be obtained as

$$X_k(z) = \sum_{l=0}^{M-1} W^{-kl} (z^{-l} E_l(z^M) X(z))$$

. This shows that the M filters can be implemented as shown in figure 8.

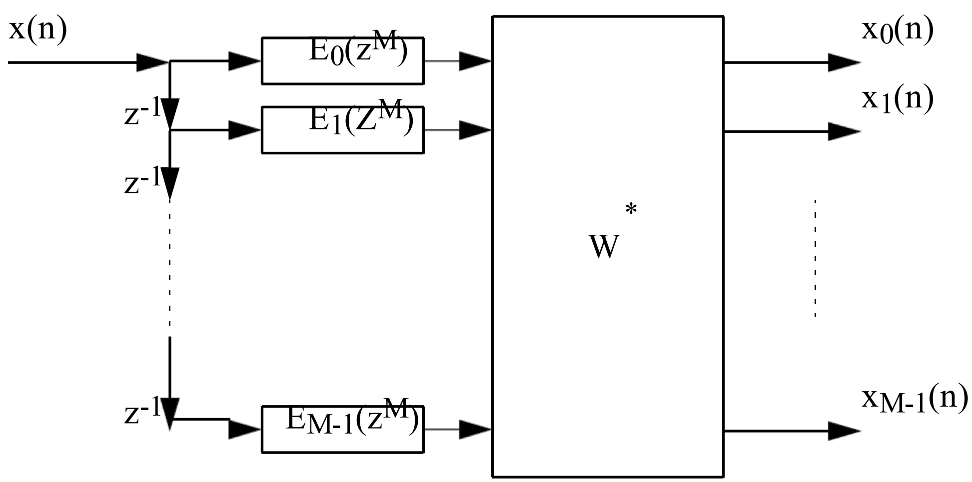


fig 9 - Implementation of the uniform DFT filter bank using polyphase decomposition.

Decimated uniform DFT banks.

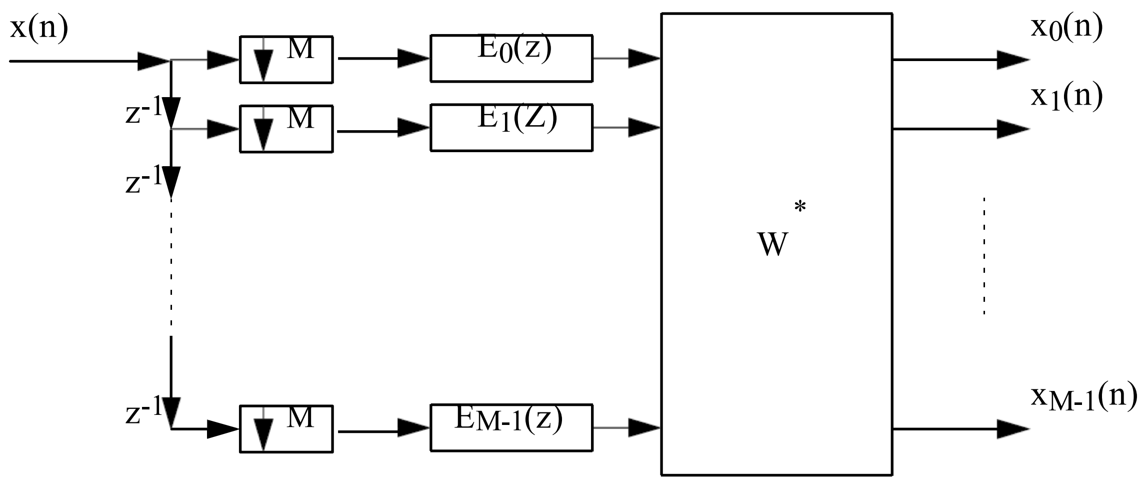


fig 10. Redrawing of fig -9 when $x_k(n)$ is decimated by M

4. Commutator models

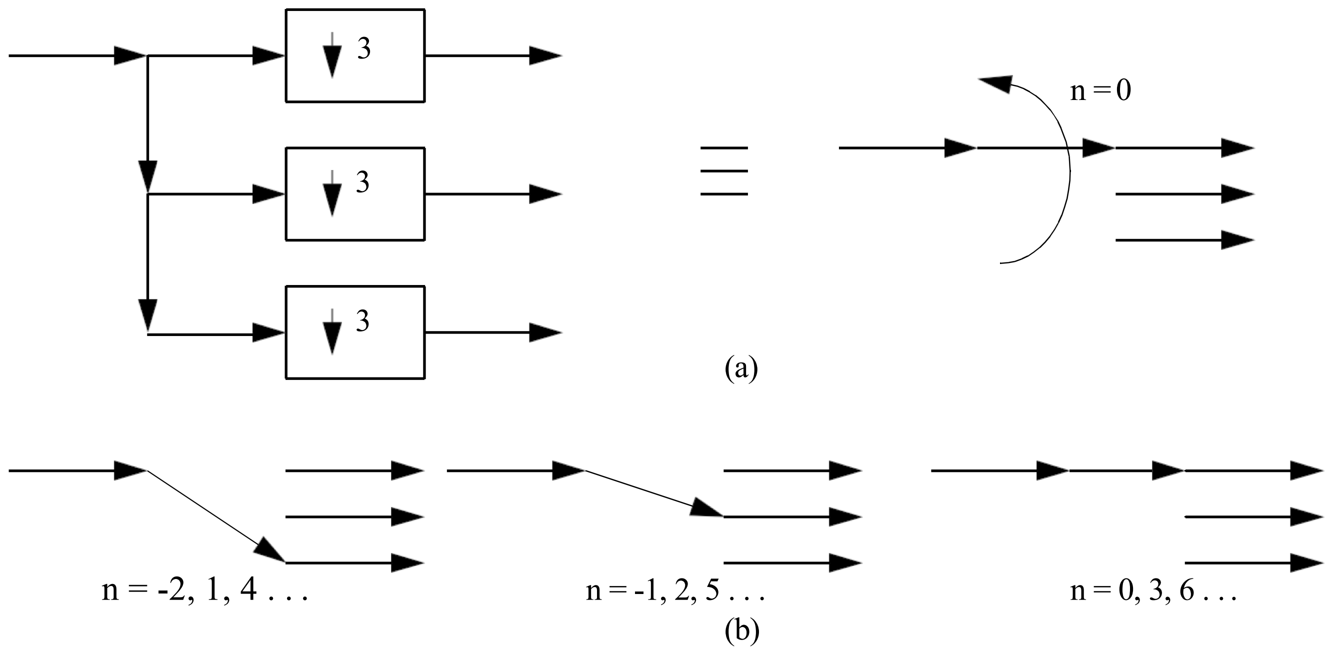


fig 11. The counterclockwise commutator model for a delay chain followed by decimators. (a) Example with M = 3, and (b) operation of the commutator switch.

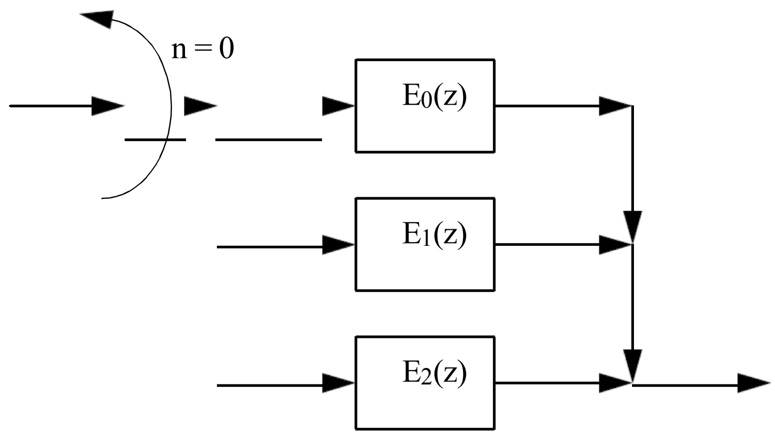


fig 12. The polyphase implementation of a decimation filter (M = 3) using a counterclockwise commutator model

5. The Polyphase Matrix

a) Analysis Bank

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M) \qquad \text{(Type 1 polyphase)} \tag{15}$$

This equation can be written in matrix form as

$$\begin{bmatrix} H_0(z) \\ \dots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z) & E_{01}(z) & E_{0,M-1}(z) \\ \dots & \dots & \dots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & E_{M-1,M-1}(z) \end{bmatrix} \begin{bmatrix} 1 \\ \dots \\ z^{-(M-1)} \end{bmatrix} \tag{16}$$

that is, as

$$h(z) = E(z^M)e(z) \tag{17}$$

where $$\mathrm{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) & E_{0,M-1}(z) \\ \dots & \dots & \dots \\ E_{M-1,0}(z) & E_{M-1,1}(z) & E_{M-1,M-1}(z) \end{bmatrix} \tag{18}$$

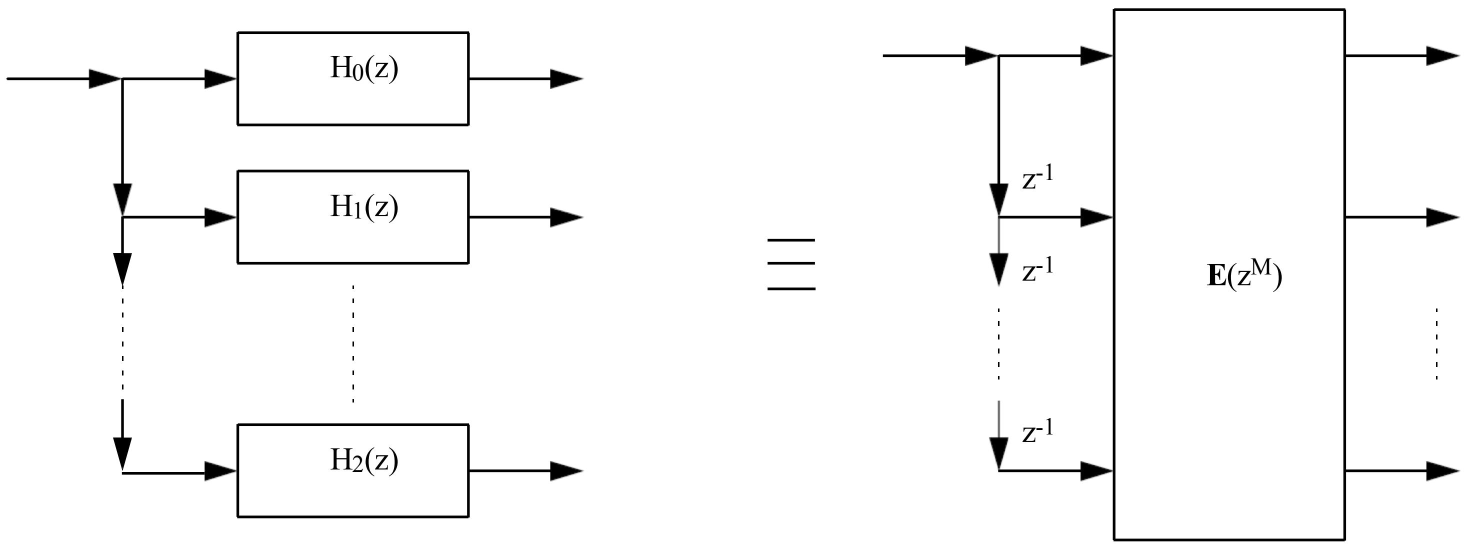


fig 13. Type1 polyphase representation of an analysis bank. E(z) is called the polyphase component matrix for the analysis bank.

b) Synthesis Bank

The set of synthesis filters can also expressed in a similar manner as

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{lk}(z^M) \qquad \text{(Type 2 polyphase)} \tag{19}$$

using matrix notation we have

$$\begin{bmatrix} F_0(z) & \dots & F_{M-1}(z) \end{bmatrix} = \begin{bmatrix} z^{-(M-1)} & \dots & 1 \end{bmatrix} \begin{bmatrix} R_{00}(z^M) & \dots & R_{0,M-1}(z^M) \\ R_{10}(z^M) & \dots & R_{1,M-1}(z^M) \\ R_{M-1,0}(z^M) & \dots & R_{M-1,M-1}(z^M) \end{bmatrix} \tag{20}$$

this can be written as

$$f^T(z) = z^{-(M-1)} \tilde{e}(z) R(z^M) \quad \text{where} \tag{21}$$

$$\mathrm{R(z)} = \begin{bmatrix} R_{00}(z) & \dots & R_{0,M-1}(z) \\ R_{10}(z) & \dots & R_{1,M-1}(z) \\ R_{M-1,0}(z) & \dots & R_{M-1,M-1}(z) \end{bmatrix} \tag{22}$$

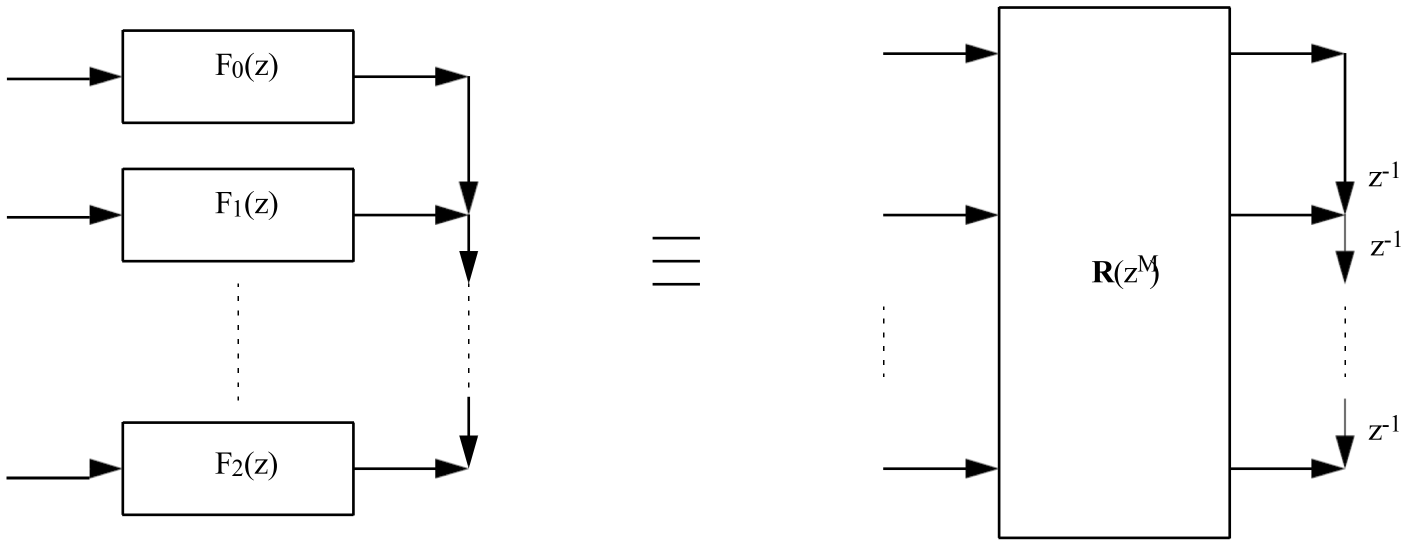


fig 14. Type 2 polyphase representation of a synthesis bank. R(z) is the polyphase component matrix for the synthesis bank.

6. Polyphase representation of M-channel QMF bank

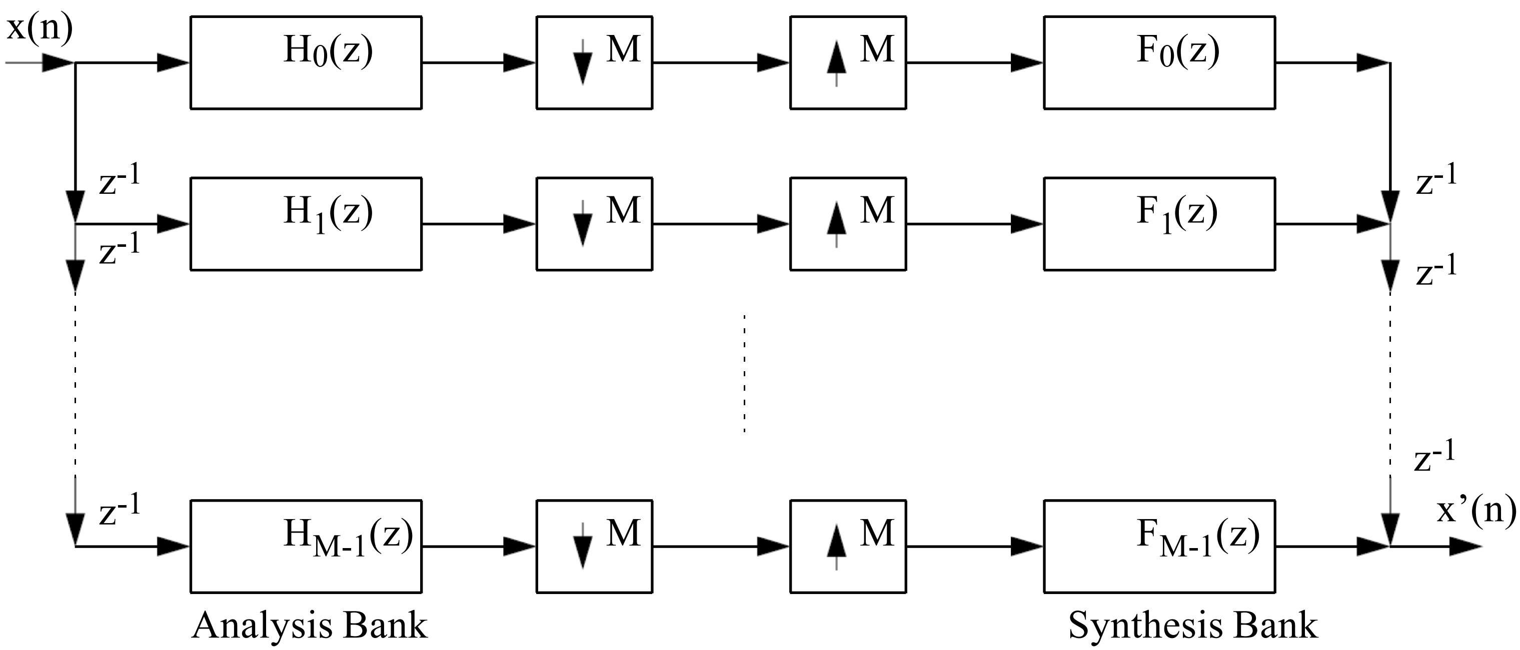


fig 15. The M-channel maximally decimated filter bank, also called M-channel QMF bank

Using the polyphase representation for analysis and synthesis bank(fig 13 and fig14), the M-channel maximally decimated filter bank can be represented as shown in fig 16

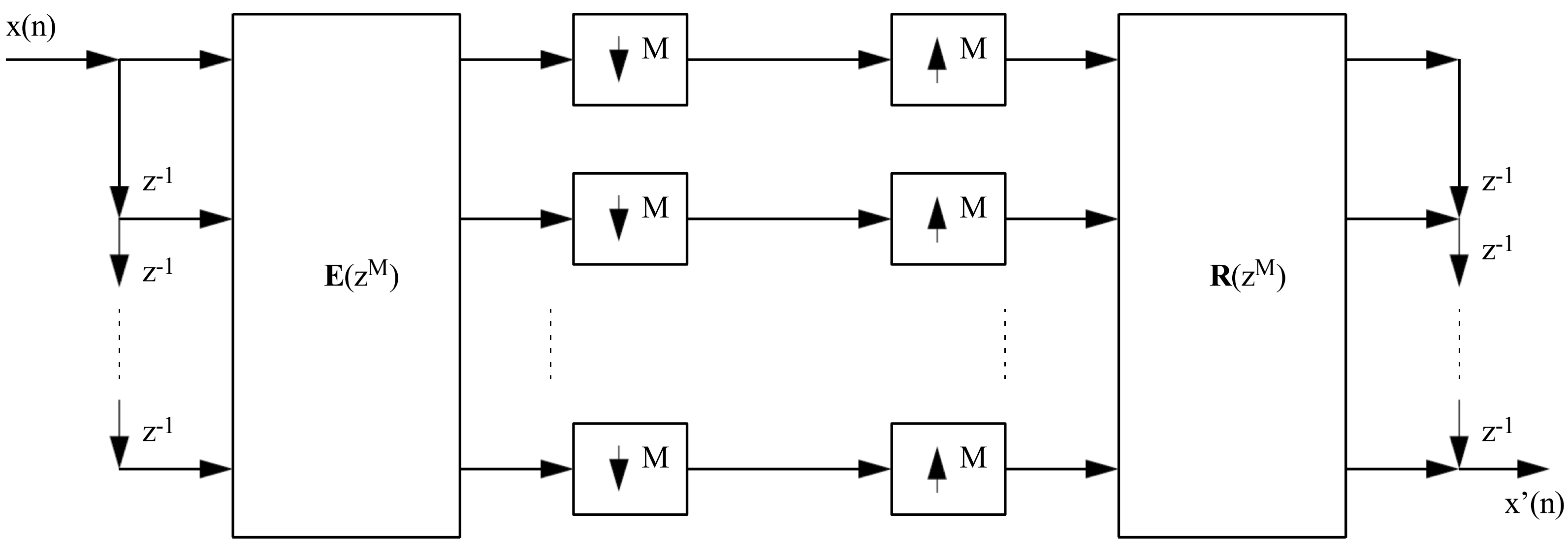


fig 16. Polyphase representation of an M-channel maximally decimated filter bank

By using noble identities, we can redraw fig 16 as shown in fig 17.

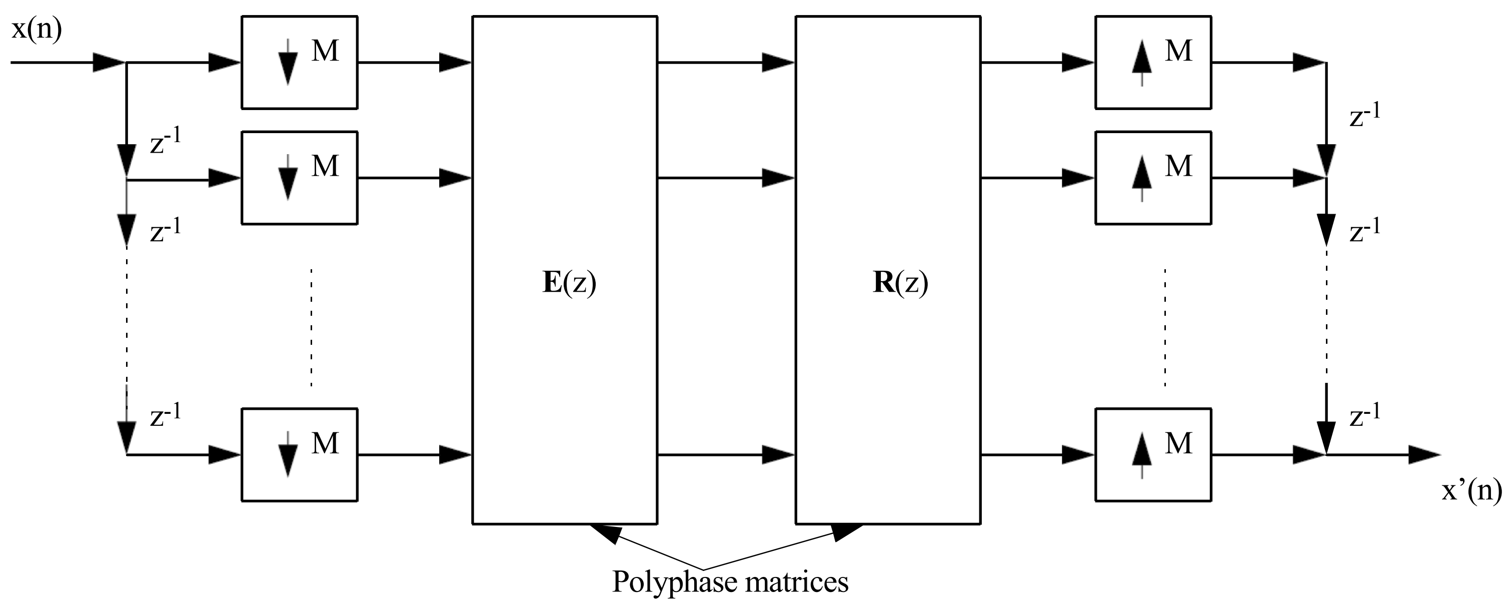


fig 17. Rearrangement of fig.16 using noble identities

Finally, we can combine the matrices and redraw the system as in fig 18.
The M x M matrix P(z) is defined as

$$P(z) = R(z)E(z)$$

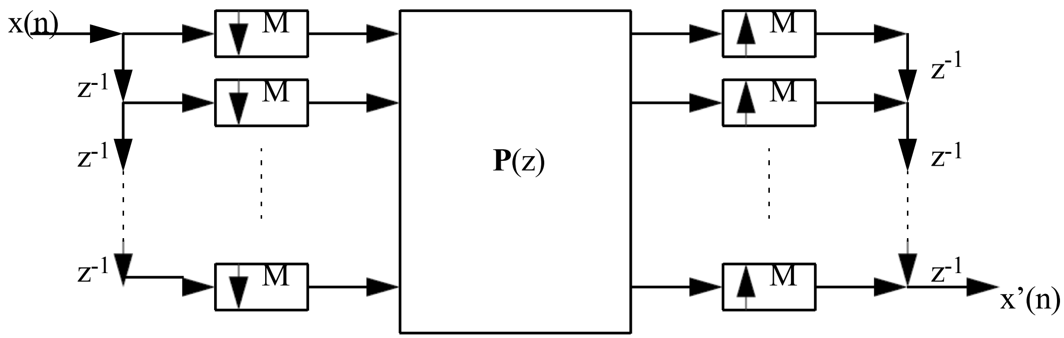


Fig 18. Simplification of fig 17 P(z) = R(z)E(z).

7. Relation between polyphase matrix and AC matrix

The Alias Component matrix(AC matrix) H(z) and the polyphase component matrix E(z) of any M-channel analysis bank are related as

$$H(z) = WD(z)E^T(z^M) \quad \text{where} \tag{23}$$

$$D(z) = diag\ 1\left[\ldots\ z^{-(M-1)}\right] \quad \text{and} \tag{24}$$

W is M x M DFT matrix.